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ON NATURAL EXACTLY COVERING SYSTEMS OF CONGRUENCES HAVING MODULI OCCURRING AT MOST M TIMES

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In this paper we generalize to NECS's(M), certain results obtained in [2] for NECS's(2), namely the boundedness of (a) the greatest prime divisor of any modulus, (b) the number of disjoint systems and of (c) the smallest modulus.

1. Introduction

Various authors (see, for example, [4,6,9]) have been concerned in the last twenty years with the subject of Covering Systems, Exactly Covering Systems and related problems.

Exactly Covering Systems, connected with certain rooted trees, whose moduli occur at most twice are discussed in [2]. In this paper, results of the same nature as those in [2], are obtained when the moduli occur at most M times. Similar results were independently obtained by Znám.

In order to make the paper self-contained, we shall introduce here some definitions, notation and known results of [2].

Definition 1.1. An *Exactly Covering System* (abbreviated ECS) is a set of ordered pairs of integers (a_i, n_i), $i = 1, 2, ..., k$, $a_i \geq 0$, $1 < n_1 \leq n_2 \leq ... \leq n_k$, such that every integer satisfies *exactly* one of the congruences $x \equiv a_i \pmod{n_i}$.

Definition 1.2. An ECS, in which for every value m there are at most M moduli which are equal to m, will be called an ECS(M).

It is known [5,4] that the moduli $n_1 \leq n_2 \leq ... \leq n_k$ of an ECS satisfy:

$$(1) \qquad \sum_{i=1}^{k} \frac{1}{n_i} = 1 ,$$

(2) $\quad n_{k-1} = n_k$.

Condition (2) implies moreover that $M = 1$ is impossible.

Definition 1.3. The largest prime that divides any modulus of any ECS(M), will be denoted by $P(M)$.

Definition 1.4. ECS's(M) will be called *disjoint*, if in the set containing all their moduli, every modulus occurs at most M times.

Definition 1.5. The maximal number of disjoint ECS's(M) will be denoted by $D(M)$.

The connection between ECS's and undirected rooted trees was first shown by Znám. In [10], he used undirected rooted trees of a certain type described in the following definition.

Definition 1.6. An undirected rooted tree will be called a Z rooted tree (Z for Znám), if it does not contain any vertex of degree 2, except (possibly) for the root, the degree of which is greater or equal to 2. Natural numbers are assigned as the values of the edges. All edges incident with the root are of value equal to the degree of the root. Any other edge is of value A, where $A + 1$ is the degree of that endpoint of this edge which is closer to the root.

The product of the values of the edges of the path between any vertex V and the root V_0 is called the product-distance of V from V_0.

Connecting Z rooted trees with ECS's, Znám proved in [10] the following theorem:

Theorem 1.7 (Znám [10]). *If $n_1, ..., n_k$ are the product-distances of vertices of degree one in a Z rooted tree, then $n_1, ..., n_k$ are the moduli of an ECS.*

The converse of the theorem is false, as shown by the following example which is due to Porubsky [8]:

Example 1.8. The natural numbers

(3) $\quad \{6, 10, 15 \text{ and twenty times } 30\}$

are the moduli of an ECS(20), but are not the product-distances of vertices of degree one in any Z rooted tree.

Definition 1.9. An ECS, whose moduli $n_1, \ldots, n_k$ are the product-distances of the vertices of degree one in a Z rooted tree, will be called (after Porubsky [8]) a *Natural ECS* (abbreviated NECS).

Definition 1.10. A NECS, in which for every value m there are at most M moduli which are equal to m, will be called a NECS(M).

Definition 1.11. The largest value of an edge that can occur in the Z rooted tree corresponding to any NECS(M), will be denoted by $A(M)$.

Notation 1.12. Throughout this paper, $2 = p_1, p_2, p_3, \ldots$ will denote the consecutive primes.

Let $q_1 < q_2 < \ldots < q_s$ denote any finite sequence of distinct primes, not necessarily consecutive. We shall use quite frequently the product $\Pi_{j=1}^{s} q_j/(q_j - 1)$.

Remark 1.13. The product $\Pi_{j=1}^{s} q_j/(q_j - 1)$ is the sum of the reciprocals of all integers having only the primes $q_1, q_2, \ldots, q_s$ as prime divisors.

We can now state more precisely, the main results of this paper as follows:

Let $M \geqslant 2$ be any given value. Then

(a) The greatest value of an edge in the Z rooted tree corresponding to any NECS(M) is bounded (Theorem 2.5).

(b) The greatest prime dividing any modulus of any NECS(M) is bounded (Theorem 2.7).

(c) The maximal number of disjoint NECS's(M) is bounded (Theorem 3.1).

(d) In every NECS(M), the smallest modulus is bounded (Theorem 4.1).

2. On the greatest prime divisor of any modulus of any NECS(M)

In this section, we shall establish (Theorem 2.7) the boundedness of the greatest prime divisor $P(M)$ of any modulus of any NECS(M).

First, we shall prove two auxiliary lemmas. Secondly, the boundedness of the largest value $A(M)$ of an edge that can occur in the Z rooted tree corresponding to any NECS(M) will be established (Theorem 2.5). Theorem 2.7 will then follow as a consequence of Theorem 2.5.

Lemma 2.1. *Let $2 = p_1, p_2, p_3, \ldots$ denote the consecutive primes. Then, for any given value M ($M \geqslant 2$), there exists some value r, such that*

$$M \prod_{i=1}^{r} \frac{p_i}{p_i - 1} < p_r . \tag{4}$$

Proof. Clearly (4) is equivalent to

$$\frac{1}{p_r} \prod_{i=1}^{r} \frac{p_i}{p_i - 1} < \frac{1}{M} . \tag{5}$$

But, since by Mertens' theorem (see [7, p. 351]) the left-hand side of (5) tends to zero when r tends to infinity, it follows that for any given value M, there exists some value r, such that (5) holds.

Lemma 2.1 implies the existence of a value r satisfying (4). Hence, there also exists a minimal such value r. This justifies the following definition:

Definition 2.2. Denote by $k(M)$ the *minimal natural number*, such that

$$M \prod_{i=1}^{k(M)} \frac{p_i}{p_i - 1} < p_{k(M)} . \tag{6}$$

Remark 2.3. It is easily verified that $M > M'$ implies $k(M) \geqslant k(M')$ and $k(2) = 5$.

Lemma 2.4. *Let $2 = p_1, p_2, p_3, \ldots$ denote the consecutive primes. Then, for every fixed value M ($M \geqslant 2$), and $\alpha = 0, 1, 2, \ldots$, we have*

$$M \prod_{i=1}^{k(M)+\alpha} \frac{p_i}{p_i - 1} < p_{k(M)+\alpha} . \tag{7}$$

Proof. The proof will be done by induction on α.

Let $M \geqslant 2$ be any given value. Then, for $\alpha = 0$, the assertion is true by (6). Assume that (7) is true for $\alpha = \alpha_0 \geqslant 0$, and we shall prove it is true for $\alpha = \alpha_0 + 1$. For simplicity, we shall denote $k(M) + \alpha_0$ by t. Now, by our inductive assumption, we have

$$M \prod_{i=1}^{t} \frac{p_i}{p_i - 1} < p_t$$

and we must show that

$$M \prod_{i=1}^{t} \frac{p_i}{p_i - 1} \cdot \frac{p_{t+1}}{p_{t+1} - 1} < p_{t+1} \tag{8}$$

holds.

It suffices to show that

$$p_t \cdot \frac{p_{t+1}}{p_{t+1} - 1} \leqslant p_{t+1} \,. \tag{9}$$

But (9) is a consequence of $p_t + 1 \leqslant p_{t+1}$. This proves (8) and hence (7).

Theorem 2.5. *The largest value $A(M)$ of any edge that can occur in the Z rooted tree corresponding to any* NECS(M), *satisfies $A(M) \leqslant p_{k(M)} - 2$.*

Proof. We shall assume that for some NECS(M), the corresponding Z rooted tree G contains an edge whose value $A(M)$ is $A(M) \geqslant p_{k(M)} - 1$ and reach a contradiction.

Let G be a rooted tree of the required kind. Let V be a vertex whose degree is maximal in G, such that $A(M)$ edges stem out from, each being of value $A(M)$ and $A(M) \geqslant p_{k(M)} - 1$. Denote the respective endpoints of these $A(M)$ edges by $V_1, V_2, \ldots, V_{A(M)}$.

Suppose there are r vertices $w_1, w_2, \ldots, w_r$ of degree one in the Z rooted trees attached to the vertices $V_1, V_2, \ldots, V_{A(M)}$. If w_b $(1 \leqslant b \leqslant r)$ belongs to the Z rooted tree that stems out from the vertex V_a $(1 \leqslant a \leqslant A(M))$, then the product of the values of the edges of the path between V_a and w_b will be denoted by N_b. Let $1 \leqslant M_1 < M_2 < \ldots < M_s$ denote the different values that N_b $(b = 1, 2, \ldots, r)$ assumes. The value $M_1 = 1$ is not excluded, since $V_a = w_b$ is possible. Furthermore, each value M_j $(j = 1, 2, \ldots, s)$ occurs among the N_b's at most M times by the assumption that G corresponds to some NECS(M). Clearly, then, we have

$$\sum_{b=1}^{r} \frac{1}{N_b} = \sum_{j=1}^{s} \frac{x_j}{M_j} = A(M) \,, \qquad 1 \leqslant x_j \leqslant M, \; j = 1, 2, \ldots, s \,, \tag{10}$$

where x_j denotes the number of occurrences of M_j.

Suppose

$$M_j = \prod_{u=1}^{t} q_u^{\alpha_{ju}}, \quad q_1 < q_2 < \ldots < q_t, \quad j = 1, 2, \ldots, s, \tag{11}$$

where q_u, $u = 1, 2, \ldots, t$, are primes not necessarily consecutive. By the maximality of $A(M)$, we have $q_t \leqslant A(M)$ and by our assumption $A(M) \geqslant p_{k(M)} - 1$.

Now, we conclude our proof by distinguishing the following two cases, namely:

(i) $q_t \geqslant p_{k(M)}$,

(ii) $q_t \leqslant p_{k(M)-1}$.

Suppose (i) holds. Then, for $p_{k(M)} \leqslant q_t = p_{k(M)+\alpha}$, we obtain by (10), (11), Remark 1.13 and (7),

$$q_t \leqslant A(M) = \sum_{j=1}^{s} \frac{x_j}{M_j} \leqslant \sum_{j=1}^{s} \frac{M}{M_j} = M \sum_{j=1}^{s} \frac{1}{M_j} < M \prod_{u=1}^{t} \frac{q_u}{q_u - 1}$$

$$\leqslant M \prod_{i=1}^{k(M)+\alpha} \frac{p_i}{p_i - 1} < p_{k(M)+\alpha} = q_t,$$

a contradiction. For (ii), it follows by our assumption, (10), (11), Remark 1.13 and (6) that

$$p_{k(M)} - 1 \leqslant A(M) = \sum_{j=1}^{s} \frac{x_j}{M_j} \leqslant \sum_{j=1}^{s} \frac{M}{M_j} = M \sum_{j=1}^{s} \frac{1}{M_j} < M \prod_{u=1}^{t} \frac{q_u}{q_u - 1}$$

$$\leqslant M \prod_{i=1}^{k(M)-1} \frac{p_i}{p_i - 1} < p_{k(M)} - 1.$$

The last contradiction, taking into account the case $M_1 = 1$ and $x_1 = A(M)$, completes the proof.

Remark 2.6. The bound obtained in Theorem 2.5 is not best possible. Since $p_{k(2)} = 11$ (Remark 2.3), Theorem 2.5 implies that in every NECS(2) we have $A(2) \leqslant 9$, whereas in [2, Section 2, Lemma 2], it is shown that in every NECS(2) we have $A(2) \leqslant 8$. Moreover, this bound is actually attained, since the author possesses an example of a NECS(2) with $A(2) = 8$ (see [1, p.45]).

The main result of this section may now be stated as follows:

Theorem 2.7. *The largest prime $P(M)$ dividing any modulus of any* NECS(M), *satisfies*

$$P(M) \leqslant p_{k(M)-1}\,. \tag{12}$$

Proof. The moduli of any NECS(M) being products of values of edges in the corresponding Z rooted tree, are composed of factors which by Theorem 2.5 are bounded by $p_{k(M)} - 2$, and hence

$$P(M) \leqslant p_{k(M)} - 2. \tag{13}$$

In order to prove (12), assume the contrary $P(M) > p_{k(M)-1}$. Then, by (13) and Remark 2.3, it follows that

$$p_{k(M)-1} < P(M) \leqslant p_{k(M)} - 2 < p_{k(M)}\,.$$

This contradiction completes our proof.

Remark 2.8. The bound $p_{k(M)-1}$ obtained in Theorem 2.7, is best possible in the cases $M = 2,3$. This follows by showing first that $k(2) = 5$, $k(3) = 7$, from which $p_{k(2)-1} = 7$, $p_{k(3)-1} = 13$, and secondly, that there exists (i) a NECS(2) with $P(2) = p_{k(2)-1} = 7$ (see [2, Section 3, Example 6]), and (ii) the author possesses an example of a NECS(3) with $P(3) = p_{k(3)-1} = 13$ (see [1, p. 76]).

3. On the number of disjoint NECS's(M)

In this section, we shall establish (Theorem 3.1) the boundedness of the maximal number $D(M)$ of disjoint NECS's(M).

Theorem 3.1. *The maximal number $D(M)$ of disjoint* NECS's(M), *satisfies $D(M) \leqslant p_{k(M)} - (M + 2)$.*

Proof. Denote by $1 < M_1 < M_2 < \ldots < M_t$ all the different moduli belonging to the $D(M)$ disjoint NECS's(M). By Theorem 2.7, we have

$$M_j = \prod_{i=1}^{k(M)-1} p_i^{\alpha_{ji}}\,, \qquad j = 1, 2, \ldots, t\,, \tag{14}$$

and by (1),

$$D(M) = \sum_{j=1}^{t} \frac{x_j}{M_j}\,, \qquad 1 \leqslant x_j \leqslant M, \quad j = 1, 2, \ldots, t\,, \tag{15}$$

where x_j denotes the number of occurrences of M_j.

Now, by (15), (14), Remark 2.13, (6) and since $M_1 > 1$, it follows that

$$D(M) = \sum_{j=1}^{t} \frac{x_j}{M_j} \leqslant \sum_{j=1}^{t} \frac{M}{M_j} = M \sum_{j=1}^{t} \frac{1}{M_j} < M\left(\prod_{i=1}^{k(M)-1} \frac{p_i}{p_i - 1} - 1\right)$$

$$< p_{k(M)} - 1 - M = p_{k(M)} - (M+1)\,.$$

This implies that $D(M) \leqslant p_{k(M)} - (M+2)$ and completes our proof.

Remark 3.2. The bound obtained in Theorem 3.1 is not best possible. Since $p_{k(2)} = 11$ (Remark 2.3), Theorem 3.1 implies that $D(2) \leqslant 7$, whereas in [2], it is shown that $D(2) = 6$ and six such systems are exhibited in Section 3.

4. The boundedness of the smallest modulus of any NECS(M)

A famous question of Erdös [4] which still unanswered, concerns the boundedness of the smallest modulus in Covering Systems. In Theorem 4.1, the boundedness of the smallest modulus in NECS's(M) is established.

Theorem 4.1. *Let $M \geqslant 2$ be any fixed value. Then, in every* NECS(M), *the smallest modulus is bounded.*

Proof. Let $m_1 < m_2 < \ldots < m_k$ denote the different moduli of any NECS(M), and by Theorem 2.7 we have

$$m_j = \prod_{i=1}^{k(M)-1} p_i^{\alpha_{ji}}, \qquad j = 1, 2, \ldots, k\,. \tag{16}$$

Let $1 = c_1 < c_2 < \ldots < c_s < c_{s+1} < \ldots$ denote the sequence of all distinct natural numbers of the form $\prod_{i=1}^{k(M)-1} p_i^{\beta_i}$, where $\beta_i = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, k(M) - 1$. Let s be chosen as the least natural number satisfying

$$\sum_{t=1}^{\infty} \frac{1}{c_t} - \sum_{t=1}^{s} \frac{1}{c_t} < \frac{1}{M}\,. \tag{17}$$

We now show that $m_1 \leqslant c_s$. For if $m_1 = c_{s+u}$ ($u = 1, 2, \ldots$), then for

$1 \leqslant x_j \leqslant M$ ($j = 1, 2, \ldots, k$), by (1), (16) and (17), we have

$$1 = \sum_{j=1}^{k} \frac{x_j}{m_j} \leqslant \sum_{j=1}^{k} \frac{M}{m_j} = M \sum_{j=1}^{k} \frac{1}{m_j}$$

$$< M\left(\sum_{t=1}^{\infty} \frac{1}{c_t} \quad \sum_{t=1}^{s} \frac{1}{c_t}\right) < M \cdot \frac{1}{M} = 1\ ,$$

a contradiction. Hence, m_1 is bounded by c_s as asserted

Remark 4.2. For any given value M, the values $k(M)$ and $p_{k(M)-1}$ defined by (6) may be obtained by numerical computation. Then, the method used in the proof of Theorem 4.1 enables us to obtain a numerical bound for the smallest modulus of every NECS(M).

Remark 4.3. The bound obtained in Theorem 4.1 is not best possible. Since $P(2) = 7$ by Remark 2.8, we obtain that $m_1 < 73$ ($c_{38} = 72$). Whereas in [2, Section 4, Theorem 4], it is shown that in every NECS(2) $m_1 < 49$.

Concluding remark. Unfortunately, the methods of this paper are not sufficient in order to obtain similar results for ECS's(M) which are not NECS's(M). There exist such ECS's(M), for instance (3) and even for $M = 2$ (see [3, Example 2, p. 372]).

The difficulty is primarily due to the fact that for such systems, it is yet undecided whether the inequality $P(M) \leqslant p_{k(M)-1}$ necessarily holds. For if it does, then all the results obtained for NECS's(M), correspond as well for ECS's(M). However, if $P(M)$ does not satisfy the above inequality, but is still bounded, similar results may be obtained by the same methods.

The following question may be raised:

Question. Is it true that the largest prime $P(M)$ that divides any modulus of any ECS(M), satisfies $P(M) \leqslant p_{k(M)-1}$?

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References

[1] N. Burshtein, Exactly covering systems of congruences, Ph.D. Thesis, University of Tel Aviv, Tel Aviv, 1974.
[2] N. Burshtein, On natural exactly covering systems of congruences having moduli occurring at most twice, J. Number Theory 8 (1976).
[3] N. Burshtein and J. Schönheim, On exactly covering systems of congruences having moduli occurring at most twice, Czechoslovak Math. J. 24 (1974) 369–372.
[4] P. Erdős, On a problem concerning congruence systems, Mat. Lapok 3 (1952) 122–128.
[5] P. Erdős, Számelméleti megjegyzések IV, Mat. Lapok 17 (1962) 228–255.
[6] A.S. Fraenkel, A characterization of exactly covering congruences, Discrete Math. 4 (1973) 359–366.
[7] G. Hardy and E. Wright, An introduction to the theory of numbers, 4th ed. (Clarendon, Oxford, 1971).
[8] S. Porubsky, Natural exactly covering systems of congruences, Czechoslovak Math. J. 24 (1974) 598–606.
[9] S. Znám, On exactly covering systems of arithmetic sequences, Math. Ann. 180 (1969) 227–232.
[10] S. Znám, On the relation between exactly covering systems and rooted trees, unpublished.